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UNIQUENESS OF DEFORMATION OF THIN-WALLED RIGID-PLASTIC CYLINDERS UNDER INTERNAL PRESSURE, TENSION AND TORQUE†

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Abstract—Uniqueness of deformation of a long thin-walled rigid-plastic cylinder is examined under three types of load combination: (i) axial tension and torque, (ii) internal pressure and torque, (iii) internal pressure and axial tension. In each case, the critical rate of hardening, above which uniqueness in guaranteed, is calculated by using Hill's[1] sufficient condition for uniqueness. The effect of torque, applied at the ends of the cylinder, on lower bounds to the critical pressure and to the critical axial-load is also discussed.

INTRODUCTION

Stability of a thin rigid-plastic cylinder under internal pressure and tension was first investigated by Swift[2]. The problem was studied again by Hillier[3] using a different formulation. Yamada and Aoki[4] considered the effect of torque (twist) on a long cylinder subjected to pressure and tension by employing the method of Hill[1]. These authors, however, did not attempt to solve the complete set of equations defining the direction of strain-rate. In fact, Hill's criterion for uniqueness requires consideration of *all* velocity fields with strain-rate parallel to **m**, the unit normal to the current yield surface. In this paper, full account is taken to the admissible velocity fields and the problem of uniqueness of deformation of a long thin rigid-plastic cylinder is re-examined for three load combinations: (i) axial tension and torque, (ii) internal pressure and torque, (iii) internal pressure and axial tension.

UNIQUENESS CRITERION

Suppose that at a time t during a process of continuing deformation, the velocity v_i and the nominal traction-rates $\dot{T}_i$ are prescribed on parts S_v and S_T, respectively, of the surface S of the body. Then, a sufficient condition to guarantee uniqueness of the subsequent incremental deformation, as obtained by Hill[1], is

$$\int_V \Delta\dot{s}_{ij}\Delta v_{j,i}\,\mathrm{d}V > 0 \tag{1}$$

where V is the current volume of the body, the prefix Δ denotes the difference of corresponding quantities in two solutions, $\dot{s}_{ij}$ is the material derivative of the nominal (Lagrangian) stress s_{ij}, measured with respect to a fixed Cartesian frame of reference x_i at the instant t, and a comma signifies differentiation with respect to x_i. For an incompressible, isotropic rigid-plastic solid, the preceding condition can be simplified to

$$\int_V (h\lambda_{ij}\lambda_{ij} - \sigma_{ij}w_{k,i}w_{j,k})\,\mathrm{d}V > 0 \tag{2}$$

where $w_i \equiv \Delta v_i$ are incompressible velocity fields vanishing on S_v and are associated with

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strain-rates $\lambda_{ij} \equiv \Delta\epsilon_{ij}$ (ϵ_{ij} is the plastic strain-rate tensor) that are either zero or parallel to m_{ij}; h is a positive scalar measure of the current rate of work-hardening. When, in addition to the prescribed nominal traction-rates on S_T and velocity on S_v, a part S_p of the surface of the body is subject to a uniform fluid pressure $p(t)$, with a given pressure-rate $\dot{p}$, the uniqueness condition (2) is modified to read (see, e.g. Miles[5]):

$$\int_V (h\lambda_{ij}\lambda_{ij} - \sigma_{ij}w_{k,i}w_{j,k})\,\mathrm{d}V - p\int_{S_p} n_i w_{i,k} w_k \,\mathrm{d}S_p > 0 \tag{3}$$

where n_i is the unit outward normal to the surface. Introducing the cylindrical polar coordinate system r, θ, z, with z-direction along the axis of the cylinder, and considering the prevailing stresses for the types of load considered, the terms $h\lambda_{ij}\lambda_{ij}$, $\sigma_{ij}w_{k,i}w_{j,k}$ and $n_i w_{i,k} w_k$ in (3) can be transformed to give

$$h\lambda_{ij}\lambda_{ij} = h\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \tag{4}$$

$$\begin{aligned}\sigma_{ij}w_{k,i}w_{j,k} = {}& \sigma_{rr}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial v}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) + \frac{\partial w}{\partial r}\frac{\partial u}{\partial z}\right] + \sigma_{\theta\theta}\left[\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)\frac{\partial v}{\partial r} + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2 + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial v}{\partial z}\right] \\ & + \sigma_{zz}\left[\frac{\partial u}{\partial z}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial v}{\partial z} + \left(\frac{\partial w}{\partial z}\right)^2\right] + \sigma_{\theta z}\left[2\frac{\partial v}{\partial r}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial r}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right]\end{aligned} \tag{5}$$

$$n_i w_{i,k} w_k = \begin{cases} -\left[\dfrac{\partial u}{\partial r}u + \left(\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}\right)v + \dfrac{\partial u}{\partial z}w\right] \text{—at the cylindrical surface} \\[2ex] \pm\left[\dfrac{\partial w}{\partial r}u + \dfrac{1}{r}\dfrac{\partial w}{\partial \theta}v + \dfrac{\partial w}{\partial z}w\right] \text{—at cylinder ends.}\end{cases} \tag{6}$$

In (4)–(6), u, v, w now represent the difference of velocity fields in two modes in the r-, θ- and the z-directions, respectively.

CYLINDER UNDER AXIAL TENSION AND TORQUE

Consider a long thin-walled cylinder subjected to an axial tensile load T and a torque M, applied at the ends. In the current state, the cylinder is assumed to have thickness t and mean radius R. The only non-vanishing stress components are

$$\sigma_{zz} = \sigma(\text{say}), \qquad \sigma_{\theta z} = \tau(\text{say}). \tag{7}$$

This state of stress satisfies the current equilibrium conditions and, since the stress is everywhere at the yield point, also satisfies the Mises yield condition expressed in the form

$$\sigma^2 + 3\tau^2 = 3k^2 \tag{8}$$

where k is the yield stress in simple shear. For the stress distribution (7), the strain-rates $\lambda_{ij} \equiv \Delta\epsilon_{ij}$ must satisfy

$$\lambda_{rr} = \lambda_{\theta\theta} = -\frac{1}{2}\lambda_{zz}; \quad \lambda_{r\theta} = \lambda_{rz} = 0; \quad \lambda_{rr}/\lambda_{\theta z} = -\sigma/3\tau \tag{9}$$

or, expressed in terms of physical components of velocity u, v, w,

$$\begin{aligned}&\frac{\partial u}{\partial r} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{1}{2}\frac{\partial w}{\partial z}; \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0; \\ &\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0; \; \frac{\partial u}{\partial r}\bigg/\frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) = -\sigma/3\tau,\end{aligned} \tag{9a}$$

the general solution of which can be shown to be

$$u = -Ar + C(r^2 + 2z^2)\sin\theta$$
$$v = -\frac{6\alpha Arz}{R} + C(-r^2 + 2z^2)\cos\theta \qquad (10)$$
$$w = 2Az - 4Crz\sin\theta + 12C\alpha R^2\cos\theta,$$

where $\alpha = \tau/\sigma$ and A,C are arbitrary constants. All the quantities in the uniqueness condition (2) are evaluated using (4), (5), (8) and (10). The resulting expression is integrated about the mean radius R; a rearrangement of terms then gives the following inequality:

$$8A^2\left[\frac{3h}{2}(1+3\alpha^2) - \sigma\left(1+\frac{3}{2}\alpha^2\right)\right] + 8C^2\left[3h(1+3\alpha^2) - 2\sigma\left(1-\frac{2}{3}\frac{l^2}{R^2} - \frac{15}{2}\alpha^2\right)\right] > 0. \qquad (11)$$

For (11) to hold it is necessary and sufficient that

$$\left[\frac{3h}{2}(1+3\alpha^2) - \sigma\left(1+\frac{3}{2}\alpha^2\right)\right] > 0 \qquad (12)$$

$$\left[3h(1+\alpha^2) - 2\sigma\left(1-\frac{2}{3}\frac{l^2}{R^2} - \frac{15}{2}\alpha^2\right)\right] > 0. \qquad (13)$$

Obviously, (12) is the stronger condition and yields

$$h > \frac{2}{3}\sigma\frac{(1+3\alpha^2/2)}{(1+3\alpha^2)}. \qquad (12a)$$

The hardening parameter h can be expressed in terms of the 'critical subtangent' $\bar{z}$ on the generalized stress-strain curve (see e.g. Hillier [3]):

$$h = \frac{2}{3}\frac{d\bar{\sigma}}{d\bar{e}} = \frac{2}{3}\frac{\bar{\sigma}}{\bar{z}} = \frac{2}{\sqrt{3}}\frac{k}{\bar{z}}, \qquad \frac{1}{\bar{z}} = \frac{1}{\bar{\sigma}}\frac{d\bar{\sigma}}{d\bar{e}}, \qquad (14)$$

where the generalized stress $\bar{\sigma}$ and the generalized strain $\bar{e}$ are defined by relations

$$\bar{\sigma} = \left(\frac{3}{2}\sigma'_{ij}\sigma'_{ij}\right)^{1/2}, \quad \bar{e} = \left(\frac{2}{3}e_{ij}e_{ij}\right)^{1/2}.$$

Then, using (8) and (14), (12a) can be alternatively expressed as

$$\frac{1}{\bar{z}} > \frac{(1+3\alpha^2/2)(1-\tau^2/k^2)^{1/2}}{(1+3\alpha^2)} \qquad (12b)$$

For simple tension alone (i.e. $\tau = 0$), $1/\bar{z} > 1$. For any $\tau > 0$, the value of $1/\bar{z}$ above which uniqueness can be guaranteed is reduced; a similar conclusion was also reached by Yamada and Aoki [4]. The fact that the presence of a twist increases the magnitude of the critical subtangent, and hence the effective stress, does not however explain what happens to the axial load carrying capacity of the tube. To investigate this, consider a material model of the Ramberg–Osgood type:

$$k = k_0\bar{e}^m, \quad 0 < m < 1, \qquad (15)$$

where k_0 and m are material constants. Using (15), (14) can be written as

$$h = \frac{2}{3}\frac{k_0^{1/m}}{k^{(1-m)/m}}. \qquad (14a)$$

With the help of (12a), (14a) and the yield condition (8), the following expression is obtained for the critical axial stress:

$$\left(\frac{\sigma}{\sqrt{3}}\right)^{1/m} = \frac{mk_0^{1/m}}{(1+3\alpha^2/2)(1+3\alpha^2)^{(1-m)/2m}}. \tag{16}$$

Now, comparing the values of the axial stress for $\alpha \neq 0$ and $\alpha = 0$, denoted by $\tilde{\sigma}$ and σ_0, respectively, one gets from (16)

$$\left(\frac{\tilde{\sigma}}{\sigma_0}\right) = \frac{1}{(1+3\alpha^2/2)^m(1+3\alpha^2)^{(1-m)/2}}. \tag{17}$$

For the range $0 < m < 1$, $\tilde{\sigma}$ is always less than σ_0; the magnitude of the critical axial stress is therefore reduced. Hence, the conclusion drawn in [4] that "the inclusion of the shear stress τ tends to increase $\bar{z}$ and this makes instability less likely" seems to be incorrect.

CYLINDER UNDER INTERNAL PRESSURE AND TORQUE

Consider a closed-ended long thin-walled cylinder subjected to internal pressure p and torque M applied at its ends. At the current instant, the stress-distribution can be taken as

$$\sigma_{rr} \simeq 0; \qquad \sigma_{\theta\theta} = \frac{pR}{t}; \qquad \sigma_{zz} = \frac{pR}{2t}; \qquad \sigma_{\theta z} = \tau(\text{say}). \tag{18}$$

For the stress-state (18), the Mises yield condition reduces to

$$\frac{1}{2}\sigma_{\theta\theta}^2 + 2\tau^2 = 2k^2, \tag{19}$$

where the constant k is again the yield stress in simple shear, and the strain-rate components λ_{ij} satisfy following relations:

$$\lambda_{zz} = 0; \lambda_{rr} + \lambda_{\theta\theta} = 0; \qquad \lambda_{rz} = \lambda_{r\theta} = 0; \qquad \lambda_{rr}/\lambda_{\theta z} = -\sigma_{\theta\theta}/2\tau, \tag{20}$$

or, expressed in terms of physical components of velocity,

$$\frac{\partial w}{\partial z} = 0; \qquad \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0; \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0;$$
$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0; \qquad \frac{\partial u}{\partial r}\Big/\frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) = -\sigma_{\theta\theta}/2\tau, \tag{20a}$$

the general solution of which can be shown to be

$$u = \frac{A}{r}; \qquad v = \frac{4\beta}{R^3}Arz; \quad w = 0, \tag{21}$$

where $\beta = \tau/\sigma_{\theta\theta}$ and A is an arbitrary constant. For the velocity field (21) and the stress-state (18), the uniqueness criterion (3) is very much simplified. After using the relevant terms from (4)–(6) and then successively integrating along the z-axis, circumferentially and across the thickness t, (3) yields a simple inequality of the form

$$A^2[h(1+4\beta^2) - \sigma_{\theta\theta}(1+2\beta^2)] > 0 \tag{22}$$

which, for arbitrary A, implies that

$$h > \frac{\sigma_{\theta\theta}(1+2\beta^2)}{(1+4\beta^2)} \tag{22a}$$

for the deformation to be unique. Using (14) and (19), the inequality (22a) can also be expressed as

$$\frac{1}{\bar{z}} > \frac{\sqrt{3}(1+2\beta^2)(1-\tau^2/k^2)^{1/2}}{(1+4\beta^2)}. \tag{22b}$$

When $\tau = 0$, and hence $\beta = 0$, (22b) gives $1/\bar{z} > \sqrt{3}$, which is the well-known result for an internally pressurized long thin tube with closed ends. As in the previous section, it can be concluded that the effect of a torque is to increase the magnitude of the critical subtangent and hence the effective stress. A similar conclusion was drawn in[4] also. However, to study the effect of a twist (torque) on the magnitude of the critical pressure, consider, for example, a material model expressed by the relation (15). Then, using (14a), (19) and (22a), the following expression is obtained for the critical circumferential stress:

$$\left(\frac{\sigma_{\theta\theta}}{\sqrt{2}}\right)^{1/m} = \frac{(1/\sqrt{3})mk_0^{1/m}}{(1+4\beta^2)(1+2\beta^2)^{(1-m)/2m}}. \tag{23}$$

Let $\bar{p}$ and p_0 be the values of the critical internal pressure for $\beta \neq 0$ and $\beta = 0$, respectively. Since $\sigma_{\theta\theta} = pR/t$, it follows from (23) that

$$\frac{\bar{p}}{p_0} = \frac{1}{(1+4\beta^2)^m(1+2\beta^2)^{(1-m)/2}} \tag{24}$$

which means that for $0 < m < 1$, $\bar{p} < p_0$. In other words, the inclusion of twist results in a reduction of the critical internal pressure.

CYLINDER UNDER INTERNAL PRESSURE AND TENSION

In this section, a closed-ended long thin-walled cylinder subjected to an axial tensile load T and internal pressure p is considered. For the uniform mode of deformation, the state of stress in the current configuration is represented by

$$\sigma_{rr} \simeq 0; \qquad \sigma_{\theta\theta} = \frac{pR}{t}; \qquad \sigma_{zz} = \frac{pR}{2t} + \frac{T}{2\pi Rt} \tag{25}$$

and is assumed to satisfy the von Mises yield condition. The deviatoric stresses are

$$\sigma'_{rr} = -\frac{pR}{2t} - \frac{T}{6\pi Rt}; \quad \sigma'_{\theta\theta} = \frac{pR}{2t} - \frac{T}{6\pi Rt}; \quad \sigma'_{zz} = \frac{T}{3\pi Rt}. \tag{25a}$$

Since the components of strain-rate should be proportional to the corresponding components of deviatoric stress, one must have

$$\begin{aligned} &\lambda_{r\theta} = \lambda_{rz} = \lambda_{\theta z} = 0 \\ &\frac{\lambda_{rr}}{\lambda_{zz}} = \frac{-pR/2t - T/6\pi Rt}{T/3\pi Rt} = -\frac{1}{2}(3\gamma+1) \\ &\frac{\lambda_{\theta\theta}}{\lambda_{zz}} = \frac{pR/2t - T/6\pi Rt}{T/3\pi Rt} = \frac{1}{2}(3\gamma-1) \end{aligned} \tag{26}$$

in which $\gamma = p\pi R^2/T$. The general solution for the velocity field, u, v, w satisfying (26) is found to be (excluding rigid-body rotations and translations)

$$u = A\left(3\gamma\frac{R^2}{r} - r\right) - D[r^2 + 2z^2 + 6\gamma R^2(\log r - 1)]\cos\theta$$

$$v = D[-r^2 + 2z^2 + 6\gamma R^2 \log r] \sin\theta \tag{27}$$
$$w = 2Az + 4Drz \cos\theta$$

with A and D as arbitrary constants. Evaluating the various terms in the uniqueness condition (3) by using the stress-distribution (25) and the velocity field (27) and then performing the necessary integration yields:

$$h(1+3\gamma^2)(A^2+6D^2) - T/3\pi Rt[(1-3\gamma^2+9\gamma^3)A^2 + (27\gamma^3 - 18\gamma^2 + 6 - 4l^2/R^2)D^2] > 0 \tag{28}$$

for (28) to hold for all A,D, it is necessary and sufficient that

$$h(1+3\gamma^2) - T/3\pi Rt(1-3\gamma^2+9\gamma^3) > 0 \tag{28a}$$

$$6h(1+3\gamma^2) - T/3\pi Rt(6 - 4l^2/R^2 - 18\gamma^2 + 27\gamma^3) > 0. \tag{28b}$$

Evidently, inequality (28a) is the critical one which is associated with the uniform mode of deformation. The quantity $\bar{\sigma}$, the generalized stress, in the present case is

$$\bar{\sigma} = (\sigma_{zz}^2 + \sigma_{\theta\theta}^2 - \sigma_{zz}\sigma_{\theta\theta})^{1/2} = \frac{T}{2\pi Rt}(1+3\gamma^2)^{1/2}. \tag{29}$$

Using (14) and (29), (28a) reduces to

$$\frac{1}{\bar{z}} > \frac{(1-3\gamma^2+9\gamma^3)}{(1+3\gamma^2)^{3/2}}. \tag{30}$$

If the substitution $\alpha = 1/3\gamma \equiv T/3p\pi R^2$ is made, (30) becomes

$$\frac{1}{\bar{z}} > \frac{\sqrt{3}(1-\alpha+3\alpha^3)}{(1+3\alpha^2)^{3/2}} \tag{30a}$$

which is the same as that obtained in the limit from the corresponding result for a thick-walled cylinder[6]. The result (30a) differs slightly from that obtained by Yamada and Aoki[4]. The inequality (35) in[4] can be shown to be an immediate consequence of the inequality (3) of this paper, if the second part of the surface integral (arising from the cylinder ends) in (6) (of this paper) is zero. Since in[4], the normal component of the velocity is specified at the cylinder ends, the end-surface integral is necessarily zero. The present treatment is obviously more general because no kinematic boundary conditions are imposed.

When $\alpha = 0$ (i.e. zero tension) and $\alpha = \infty$ (i.e. zero pressure), (30a) yields familiar results

$$\frac{1}{\bar{z}} > \sqrt{3}; \quad \frac{1}{\bar{z}} > 1$$

for the expansion of a closed-ended tube under pressure and the extension of a bar under axial tension, respectively.

CONCLUSIONS

Uniqueness of deformation of a thin-walled rigid-plastic cylinder has been examined under internal pressure, tension and torque by using Hill's sufficient condition for uniqueness. It has been shown that a torque, applied at the ends of a cylinder subjected to axial tension or internal pressure, reduces the lower bound to the axial load carrying capacity or the maximum pressure.

The results of this investigation can be obtained more simply by the method of Swift[2] when it is known in advance that the uniform homogeneous mode of deformation is the critical one for non-uniqueness. However, as shown here, this information is available only through Hill's method which considers all admissible modes of deformation.

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